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Using response surface approximations in fuzzy set based design optimization*

G. Venter and R.T. Haftka

Abstract The paper focuses on modelling uncertainty typical of the aircraft industry. The design problem involves maximizing a safety measure of an isotropic plate for a given weight. Additionally, the dependence of the weight on the level of uncertainty, for a specified allowable possibility of failure, is also studied. It is assumed that the plate will be built from future materials, with little information available on the uncertainty. Fuzzy set theory is used to model the uncertainty. Response surface approximations that are accurate over the entire design space are used throughout the design process, mainly to reduce the computational cost associated with designing for uncertainty. All of the problem parameters are assumed to be uncertain, and both a yield stress and a buckling load constraint are considered. The fuzzy set based design is compared to a traditional deterministic design that uses a factor of safety to account for the uncertainty. It is shown that, for the example problem considered, the fuzzy set based design is superior. Additionally, the use of response surface approximations results in substantial reductions in computational cost, allowing the final results to be presented in the form of design charts.

1 Introduction

In the aircraft industry, structures are often designed that will be built well into the future from materials available then, leading to uncertainty in material properties. Apart from the uncertain material properties, the manufacturing cost is also uncertain. However, unlike the uncertainty in the material properties, the designer has some control over the manufacturing cost, which is closely linked to the required tolerances in geometry. For such design problems, little information regarding the uncertainty is known, and the uncertainty is typically modeled based on expert opinion and assumptions made by the designer. Fuzzy set theory can use limited available data and caters for worst case scenarios. Fuzzy set theory is thus capable of by compensating for the fact that the uncertainty is modeled based on subjective opinions and assumptions (Maglaras *et al.* 1997). In contrast, probabilistic methods require large amount of data and the results obtained are, in some cases, very sensitive to both the accuracy of this data as well as to the assumptions made during the modeling process (e.g. Ben-Haim and Elishakoff 1990, pp. 11–32).

Fuzzy set theory was introduced by Zadeh (1965) as a mathematical tool for the quantitative modeling of uncertainty, and makes use of fuzzy numbers to represent uncertain problem parameters. The designer only needs to specify the range of uncertainty and a membership function that denotes the possibility of occurrence of an element in the specified range to represent an uncertain parameter as a fuzzy number. Membership functions are generally constructed subjectively, based on expert opinion. In recent years, fuzzy set theory has been applied to a wide range of structural optimization problems. For example, Liu and Huang (1992) performed a fatigue reliability analysis of a portal frame, Jung and Pulmano (1996) considered the optimal plastic design of a fixed-fixed beam and a portal frame, and Jensen and Sepulveda (1997) minimized the weight of a 25-bar transmission tower. Fuzzy set theory has also been used in multidisciplinary optimization by Rao (1993) to design the main rotor of a helicopter as well as by Wu and Young (1996) to optimize the machine room layout of a ship. Additionally, Shih and Chang (1995) applied multicriteria optimization to various truss examples, considering both weight and displacement as objectives.

Unfortunately, designing for uncertainty is computationally intensive and typically requires at least an order of magnitude more computational cost as compared to a corresponding deterministic design. In the present paper, response surface approximations are used to reduce the high computational cost associated with designing for uncertainty by using approximations that are accurate over the entire design space to replace costly finite element analyses. Response surface approximations have attracted a lot of interest from the structural optimization community in recent years, since they filter out nu-

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merical noise inherent to most numerical analysis procedures (e.g. Giunta *et al.* 1994), they provide the designer with a global perspective of the response over the entire design space (e.g. Mistree *et al.* 1994), and they enable easy integration of various software codes (e.g. Kaufman *et al.* 1996).

An isotropic plate with a change in thickness across its width is considered as a design problem. All of the problem parameters are considered to be uncertain and the objective is to maximize a safety measure of the plate for a given weight. Both deterministic and fuzzy set based designs are considered and the results are compared. The safety measure is maximized by maximizing the factor of safety in the deterministic design and by minimizing the possibility of failure in the fuzzy set based design. Finally, the dependence of the weight on the level of uncertainty associated with the key geometric parameters is presented in the form of a design chart, based on results obtained from a number of optimizations.

2 Fuzzy set theory

Fuzzy set theory presents a methodology for the mathematical modeling of uncertainty. In contrast to classical set theory where a sharp transition exists between membership and non-membership, fuzzy set theory makes use of membership functions to denote the degree to which an element belongs to a fuzzy set. A membership function assigns a grade of membership, ranging between 0 and 1, to each element of the universal set as follows

$$\mathbf{M}(x): \; X \to [0,1] \,. \tag{1}$$

In (1) $\mathbf{M}$ denotes a membership function that maps the elements of the universal set X to the real interval [0, 1]. The same symbol, a bold face capital letter, is used to denote both the fuzzy set and its membership function. Since each fuzzy set is completely and uniquely defined by only one particular membership function, no ambiguity results from the double use of the symbol.

Fuzzy sets are represented numerically by making use of α level cuts. An α level cut is defined as the real interval where the membership function is larger than a given value, α (Klir and Yuan 1995, p. 19) and may be written mathematically for a generic fuzzy set $\mathbf{B}$ as follows:

$$^{\alpha}B = \{x | \mathbf{B}(x) \geq \alpha\} \,. \tag{2}$$

Figure 1 provides a graphical representation of (2), where it is assumed that $\mathbf{B}$ has a triangular and symmetric membership function, and shows the end points $^{\alpha}b_1$ and $^{\alpha}b_2$ of the α level cut.

A fuzzy number is defined as a fuzzy set that is both normal and convex (Klir and Yuan 1995, pp. 97). A normal fuzzy set has a maximum membership function equal to 1, while all possible α level cuts are convex for a convex fuzzy set. The fuzzy set $\mathbf{B}$ shown in Fig. 1 is thus a fuzzy number. In fact the triangular and symmetric membership function is most often used to represent fuzzy numbers, mainly due to its simplicity, and was used throughout the present paper to represent all of the uncertain problem parameters.

Fig. 1. An α level cut of a triangular and symmetric membership function, having support in (x_L, x_R)

A fuzzy function $\mathbf{Y}$ is a function of fuzzy variables $\mathbf{X}_i$ and may be written as

$$\mathbf{Y} = \mathbf{Y}(\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n) \tag{3}$$

for the case where n fuzzy variables are considered. Klir and Yuan (1995, pp. 105–109) summarized and proved the following properties of a fuzzy function.

1. When all of the fuzzy variables of a fuzzy function are continuous fuzzy numbers, the fuzzy function itself is also a continuous fuzzy number.
2. When all of the fuzzy variables of a fuzzy function are fuzzy numbers, the α level cut of a fuzzy function $^{\alpha}Y$ may be written in terms of the α level cuts of its fuzzy variables $^{\alpha}X_i$ as follows:

$$^{\alpha}Y = {}^{\alpha}Y(^{\alpha}X_1, \ldots, {}^{\alpha}X_n) = \tag{4}$$
$$\left[\min_{^{\alpha}R}\left[Y(^{\alpha}X_1, \ldots, {}^{\alpha}X_n)\right], \max_{^{\alpha}R}\left[Y(^{\alpha}X_1, \ldots, {}^{\alpha}X_n)\right]\right],$$

where $^{\alpha}R$ denotes the n-dimensional box, formed by the α level cuts of the n fuzzy numbers.

Based on these properties of a fuzzy function, Dong and Shah (1987) introduced the vertex method for evaluating the upper and lower bounds of $^{\alpha}Y$ when all of the fuzzy variables of $\mathbf{Y}$ are fuzzy numbers. This method requires the evaluation of the fuzzy function at the 2^n vertices of the n-dimensional rectangle, formed by the α level cuts of the n fuzzy variables. In addition, interior global extreme points need to be checked. This method requires a large number of function evaluations and is computationally intensive.

For calculating the possibility of failure it is required to compare a crisp number with a fuzzy number. Note that a fuzzy number may also be considered as the trace of a possibility measure Π on the singletons (single elements) x of the universal set X (Dubois and Prade 1988, p. 13–17). When a possibility measure defined on the unit interval is considered, its possibility distribution π is then interpreted as the membership function of a fuzzy number **B** describing the event that Π focuses on, as follows:

$$\Pi\ (\{x\}) = \pi(x) = \mathbf{B}(x)\ , \quad \forall x \in X\ . \tag{5}$$

The possibility measure of a crisp number being smaller or equal to a fuzzy number **B** is then defined (Dubois and Prade 1988, pp. 99–101) as follows:

$$\Pi_{\mathbf{B}}\ ([x, +\infty)) = \sup_{y \geq x} \mathbf{B}(y)\ , \quad \forall x\ . \tag{6}$$

The possibility distribution function $\pi_{\mathbf{B}}$ corresponding to the possibility measure of (6) is shown graphically in Fig. 2 for the general case where **B** has a nonlinear membership function.

Fig. 2. Possibility distribution of $\mathbf{B} \geq x$ for nonlinear $\mathbf{B}(x)$, having support in (x_L, x_R)

Based on (5) and (6), the possibility distribution of failure $\pi_{(\mathbf{P}-\mathbf{P}_f)}$ is obtained from the fuzzy function $(\mathbf{P} - \mathbf{P}_f)$ that contains the fuzzy numbers **P** (the applied load) and $\mathbf{P}_f$ (the failure load) as variables. The possibility of failure $(\mathbf{P} - \mathbf{P}_f \geq 0)$ is then defined as

$$\Pi_{(\mathbf{P}-\mathbf{P}_f)}\ ([0, +\infty)) = \sup_{y \geq 0} (\mathbf{P} - \mathbf{P}_f)(y)\ . \tag{7}$$

3 Overview of response surface approximations

A response surface approximation is an approximate relationship between a dependent variable η (the response) and a vector $\mathbf{x}$ of k independent variables (the predictor variables). The response is generally obtained from experiments (which may be numerical in nature), where η denotes the mean or expected response value. It is assumed that the true model of the response may be written as a linear combination of given functions $\tilde{\mathbf{z}}$ with some unknown coefficients $\tilde{\beta}$. The experimentally obtained response y differs from the expected value η due to random experimental error δ as follows:

$$y(\mathbf{x}) = \eta(\mathbf{x}) + \delta = \tilde{\mathbf{z}}(\mathbf{x})^T \tilde{\beta} + \delta\ . \tag{8}$$

Since the exact dependence of η is generally unknown, a response surface approximation is used to approximate $\eta(\mathbf{x})$ as follows:

$$y(\mathbf{x}) = \mathbf{z}(\mathbf{x})^T \beta + \varepsilon\ , \tag{9}$$

where $\mathbf{z}(\mathbf{x})$ contains the assumed functions in the response surface approximation and β the associated coefficients. Furthermore, ε denotes the total error, which is the difference between the predicted and measured response values and includes both random (variance) and modeling (bias) error. Typically low order polynomials are used as a response surface approximation, in which case $\mathbf{z}(\mathbf{x})$ consists of monomials.

The coefficients β of the response surface approximation are estimated from the experimentally obtained response values to minimize the sum of the squares of the error terms, a process known as regression. The estimated values of β is denoted by **b**, resulting in the following response surface approximation:

$$\hat{y}(\mathbf{x}) = \mathbf{z}(\mathbf{x})^T \mathbf{b}\ , \tag{10}$$

where the caret symbol implies predicted values.

The assumed form of the response surface approximation, (10), usually includes redundant parameters and parameters that are poorly characterized by the experiments. These parameters may increase the prediction error of the approximation and thus decrease its predictive capabilities. In the present paper, redundant parameters are eliminated by using mixed, backwards, stepwise regression (e.g. Ott 1993, pp. 648–659; Myers and Montgomery 1995, pp. 642–655). Mallow's Cp statistic is used to identify the best reduced response surface approximation from the subset of reduced response surface approximations provided by the stepwise regression procedure and is defined as

$$Cp = \frac{SSE_p}{s_\varepsilon^2} - (n - 2p)\ , \tag{11}$$

where SSE_p is the sum of the squares of the n error terms (one for each data point used to estimate **b**) for an approximation with p parameters and s_ε^2 is the mean sum of squares of the error terms obtained from the response surface approximation with all of the parameters included.

Optimization has the general tendency of exploiting weaknesses in the formulation of the response function, and highly accurate response surface approximations are thus a requirement in structural optimization applications. To ensure highly accurate approximations, it is important to evaluate the predictive capabilities of the approximations. In the present paper, the coefficient of determination (R^2) statistic, the adjusted R^2 (Adj-R^2) statistic, the percent root mean square error (%RMSE) as well as the percent root mean square error based on the predicted sum of squares (PRESS) statistic (%$\text{RMSE}_{\text{PRESS}}$) are calculated (Myers and Montgomery 1995, pp. 28–47).

The R^2 statistic denotes the proportion of the variability in the response that is accounted for by the response surface approximation and has a value between 0 and 1. The Adj-R^2 statistic is an alternative measure of the explained variability that, unlike R^2, has the desirable property that its value does not necessarily increase when adding (possibly redundant) parameters to a response surface approximation. The %RMSE is an estimate of the root mean square error of the approximation that is obtained from the data points used to construct the approximation, using the following unbiased estimator:

$$\%\mathrm{RMSE} = \frac{100}{\overline{y}} = \sqrt{\frac{1}{(n-p)}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}\,,$$

where

$$\overline{y} = \frac{1}{n}\sum_{i=1}^{n}|y_i|\,. \tag{12}$$

The %RMSE$_{\mathrm{PRESS}}$ is an additional measure of the error, based on the PRESS statistic. The PRESS statistic is calculated by selecting a data point, say data point i. The response surface approximation obtained from the remaining $(n-1)$ data points is used to predict the response at the withheld data point, denoted by $\hat{y}_{(i)}$. The prediction error at the withheld data point $e_{(i)}$ is then defined as

$$e_{(i)} = y_i - \hat{y}_{(i)}\,, \tag{13}$$

and is referred to as the i-th PRESS residual. This procedure is repeated for all of the data points and the resulting PRESS residuals are summed to form the PRESS statistic as follows:

$$\mathrm{PRESS} = \sum_{i=1}^{n} e_{(i)}^2 = \sum_{i=1}^{n}\left[y_i - \hat{y}_{(i)}\right]^2\,. \tag{14}$$

The

$$\%\mathrm{RMSE}_{\mathrm{PRESS}} = \frac{100}{\overline{y}}\sqrt{\frac{1}{n}\mathrm{PRESS}}\,. \tag{15}$$

4 Plate example

An isotropic plate with a change in thickness in the form of a linear ramp (see Fig. 3) is the design problem considered in the present paper.

Three nondimensional parameters, λ, β and γ, are used to specify the geometry and location of the change in thickness (see Fig. 4). The plate is simply supported on two edges, free on the other two edges, and subjected to an uniformly distributed load applied on the two simply supported edges.

Both a yield stress failure (according to the Von Mises criterion) and a buckling load constraint are considered in the design, and the failure load P_f of the plate is calculated from

$$P_f = \min\begin{cases}\dfrac{\sigma_Y \lambda b t_0}{\tilde{\sigma}_x}\\[2mm] \dfrac{\tilde{N}_{\mathrm{crit}}\pi^2 E b(\lambda t_0)^3}{12(1-\nu^2)a^2}\end{cases}, \tag{16}$$

Fig. 3. Three-dimensional view of the plate with a thickness change.

Fig. 4. Cross-section of plate with response variables shown.

and failure is defined to occur when:

$$P - P_f \geq 0\,. \tag{17}$$

In (16) and (17), σ_y denotes the yield stress, E the Young's modulus and ν the Poisson's ratio of the material considered, while λ, a, b, t_0 and r describe the geometry of the plate as shown in Figs. 3 and 4 and P denotes the applied load. Additionally, $\tilde{\sigma}_x$ denotes the nondimensional, x-directional stress component on the top surface of the thin section of the plate, calculated a distance r from the re-entrant corner, and is defined as

$$\tilde{\sigma}_x = \frac{\lambda b t_0 \sigma_x}{P}\,, \tag{18}$$

while $\tilde{N}_{\mathrm{crit}}$ denotes the nondimensional buckling load of the plate, defined as

$$\tilde{N}_{\mathrm{crit}} = \frac{12(1-\nu^2)a^2 N_{\mathrm{crit}}}{\pi^2 E b(\lambda t_0)^3}\,. \tag{19}$$

Venter *et al.* (1997) used a large number of numerical experiments to study this problem in detail, and showed that the maximum von Mises stress always occurs on the top surface of the thin section of the plate, in which case σ_x is the only nonzero stress component. According to the von Mises criterion, failure then occurs when

$$\tilde{\sigma}_x \geq \tilde{\sigma}_Y = \frac{\lambda b t_0 \sigma_Y}{P} \,. \tag{20}$$

Venter *et al.* (1997) also determined that the problem has both a local and a global buckling mode, and defined a simple geometric criterion to distinguish between the two buckling modes as follows:

$$\text{buckling mode} = \begin{cases} \text{local if } \frac{(0.5-\beta-\gamma)}{\lambda} \geq 0.6 \\ \text{global if } \frac{(0.5-\beta-\gamma)}{\lambda} \leq 0.6 \end{cases} . \tag{21}$$

Venter *et al.* (1997) constructed highly accurate response surface approximations for both the x-directional stress distribution on the top surface of the thin section of the plate and for the buckling load of the plate, using a total of 752 finite element analyses. Numerical experiments in the form of finite element analyses were conducted using MSC\NASTRAN Version 68. A cross-section of the plate was used to model the stress distribution near the re-entrant corner, using four-node, isoparametric, plane strain elements. All of these models had a uniform mesh, with roughly 1,800 elements in the x-direction and 9 elements in the z-direction, but the number of elements varied slightly from model to model. A schematic representation of the finite element model used is shown in Fig. 5.

Fig. 5. Finite element model used for stress distribution about the re-entrant corner

For the buckling load response surface approximations, four-node, isoparametric, plate bending elements were used to construct a two-dimensional finite element model similar to a plan view of Fig. 3. Twenty elements were used in each of the x- and y-directions respectively. The eccentricity of the mid-plane was found to have an insignificant impact on the buckling load value (note that the sides of the plate are free) and was ignored in the analysis.

The stress distribution response surface approximation (see Venter *et al.* 1997) may be written in functional form as

$$\tilde{\sigma}_x = \tilde{\sigma}_x \left(\lambda, \beta, \gamma, \tilde{r}^{\zeta-1} \right) , \tag{22}$$

where ζ is a constant that describes the radial stress distribution near the re-entrant corner and depends on λ, γ and a/t_0 through the angle Θ. Additionally, $\tilde{r}$ is the nondimensional distance measured from the re-entrant corner, defined as:

$$\tilde{r} = r/t_0 \,. \tag{23}$$

Additionally, two response surface approximations, corresponding to the local and global buckling modes were constructed, which may be written in functional form as:

$$\tilde{N}_{\text{loc}} = \tilde{N}_{\text{loc}}(\lambda, \beta, \gamma) \,, \quad \tilde{N}_{\text{glob}} = \tilde{N}_{\text{glob}}(\lambda, \beta, \gamma) \,. \tag{24}$$

The design space used for constructing the stress distribution and buckling load response surface approximations is summarized in Table 1. The upper limit on $\tilde{r}$ limits the radius of the yield zone about the re-entrant corner to be no greater than 80% of the thickness of the thin section of the plate, while the upper bound on γ is dictated by the geometry of the transition region.

Table 1. Design space for constructing the response surface approximation approximations

Response variable	Range
λ	$0.2 \leq \alpha \leq 1.0$
β	$-0.475 \leq \beta \leq 0.475$
γ	$0 \leq \gamma \leq 0.475 - \beta$
$\tilde{r}^{\zeta-1}$	$0 \leq \tilde{r} \leq 0.8\alpha$

The stress distribution response surface approximation was constructed from 288 plate configurations (corresponding to 288 finite element analyses). Each plate configuration included a number of data points with different $\tilde{r}^{\zeta-1}$ values (corresponding to different finite elements), yielding a total of 2,124 data points. The buckling load response surface approximations were constructed from an additional 288 finite element analyses. Using the geometric criterion of (21), these 288 finite element analyses were divided into two groups corresponding to the two buckling modes. This process identified 126 data points for constructing the local buckling load approximation and 162 data points for constructing the global buckling load approximation. A quartic polynomial was used as initial response surface approximation for both the stress distribution and the global buckling load response surface approximations, while a cubic polynomial was used for the local buckling load response surface approximation. These initial response surface approximations were reduced, using the mixed stepwise regression procedure and the Cp statistic. The process of constructing the response surface approximations is discussed in more detail by Venter *et al.* (1997). The predictive capabilities of the reduced response surface approximations are summarized in Table 2.

5 Design problem formulation

The design problem has two objectives. The first objective is to maximize a safety measure of the plate for a given weight.

Table 2. Predictive capabilities of stress distribution and buckling load response surface approximations

Model	$\mathbf{R^2}$	Adj-$\mathbf{R^2}$	RMSE [%]	PRESS [%]
Stress	4-th order model (2,124 data points)			
Reduced 43 terms	0.9983	0.9982	3.2964	3.3886
Local buckling	3-rd order model (126 data points)			
Reduced 19 terms	0.9999	0.9998	0.5550	0.6920
Global buckling	4-th order model (162 data points)			
Reduced 25 terms	0.9910	0.9895	2.4888	3.0202

Table 3. Problem parameters and associated uncertainty

Variable	Nominal values	Level of uncertainty, $\mathbf{u}$
$\lambda^\dagger$	[0.2 – 1.0]	[± 2 – ± 20]%
$\beta^\dagger$	[-0.4 – 0.4]	[± 2 – ± 20]%
$\gamma^\dagger$	[0 – 0.8]	[± 2 – ± 20]%
a	228.6 cm	±5%
b	127.0 cm	±5%
t_0	7.620 cm	±5%
E	206.84 GPa	±5%
ν	0.29	±5%
σ_y	197.26 MPa	±10%
r	$5\alpha t_0$	±10%
P	3,224.96 kN	±10%

† Design variables

The results obtained from a traditional deterministic approach, using a factor of safety to account for the uncertainty, were compared to those obtained from a fuzzy set based approach. The safety measure of the plate was maximized, by maximizing the factor of safety for the deterministic approach and by minimizing the possibility of failure for the fuzzy set based approach. Note that there exist fundamental differences between the deterministic and fuzzy set based approaches for maximizing the safety measure of the plate for a given weight. The deterministic approach tends to equalize the failure load of each failure criterion, while the fuzzy set based design tends to equalize the possibility of failure of each failure criterion.

The second objective is to study the dependence of the weight of the final design on the level of uncertainty associated with the design variables λ, β and γ. In this case, the weight was minimized for a specified allowable possibility of failure and different levels of uncertainty associated with the design variables. The results are presented in the form of a design chart. Different levels of uncertainty for the design variables were considered, since these geometric variables have the largest influence on the manufacturing cost of the plate. If the tolerances of these variables can be relaxed without a large penalty in terms of weight, substantial cost savings can be achieved in manufacturing the plate. The problem parameters and associated levels of uncertainty used are summarized in Table 5. Although Table 5 has a total of 11 uncertain problem parameters, only 8 uncertain parameters are associated with each of the two failure criteria [see (16), (22) and (24)].

5.1 Deterministic design

The objective of the deterministic design is to maximize the factor of safety for a given weight. However, since it is difficult to specify a meaningful weight, it was decided to minimize the weight for a given factor of safety. The resulting minimum weight was then used as the given weight for the fuzzy set based design. A factor of safety of 1.5 was assumed and the level of uncertainty associated with the design variables was considered to be constant, equal to ±5%. The nondimensional cross-sectional area of the plate $\tilde{A}$ was used as a representative value of the weight and the resulting optimization problem may be written as

minimize:

$$\tilde{A} = \frac{A}{\lambda t_0} = \frac{1}{2}(1 + 2\beta + \gamma) + \frac{\lambda}{2}(1 - 2\beta - \gamma)\,,$$

subject to

$$\frac{\beta}{0.4} + 1 \geq 0\,, \quad 1 - \frac{\beta}{0.4} \geq 0\,, \quad \gamma \geq 0\,,$$
$$1 - \frac{\gamma + \beta}{0.4} \geq 0\,, \quad \frac{P_f}{P} - 1.5 \geq 0\,. \tag{25}$$

The constraints involving β and γ are geometric constraints and P_f is calculated from (16), using the nominal values of the design variables.

5.2 Fuzzy set based design

The fuzzy set based design problem minimizes the possibility of failure, using the optimum nondimensional cross-sectional area obtained from (25) as an upper limit of the weight. The resulting optimization problem may be written as

Minimize:

$$\Pi_{(\mathbf{P}-\mathbf{P}_f)} = \Pi_{(\mathbf{P}-\mathbf{P}_f)}(\boldsymbol{\lambda}, \boldsymbol{\beta}, \boldsymbol{\gamma})\,,$$

subject to

$$\frac{\beta}{0.4} + 1 \geq 0\,, \quad 1 - \frac{\beta}{0.4} \geq 0\,, \quad \gamma \geq 0\,,$$
$$1 - \frac{\gamma + \beta}{0.4} \geq 0\,, \quad \frac{\tilde{A}(\lambda, \beta, \gamma)}{\tilde{A}^*} - 1 = 0\,. \tag{26}$$

where bold face Greek symbols denote fuzzy numbers while regular font symbols denote nominal values. Additionally, $\Pi_{(\mathbf{P}-\mathbf{P}_f)}$ denotes the possibility of failure and $\tilde{A}^*$ denotes the optimum nondimensional cross-sectional area obtained from the deterministic design of (25).

5.3 Implementation of the fuzzy set based design

In the present work response approximations form an integral part of the fuzzy set based design and two levels of response surface approximations are employed during the different stages of the design process. On the first level, the stress distribution and buckling load response surface approximations (Section 4) are used to replace computationally expensive finite element analysis in evaluating the possibility of failure. The possibility of failure is calculated from (16), using the vertex method. When considering all of the problem parameters as uncertain, the evaluation of the possibility of failure for a single α level cut value requires $2 \times 2^8 = 512$ (recall that each failure criterion has a total of 8 uncertain problem parameters) finite element analyses when no response surface approximations are used. In terms of a single optimization, an estimate of the required number of finite element analyses required when not using response surface approximations, is obtained from the product of four numbers as follows:

Average number of design optimization iterations:	5
Average number of $\Pi_{(\mathbf{P}-\mathbf{P}_f)}$ evaluations per iteration:	6
Average number of α level cut evaluations per $\Pi_{(\mathbf{P}-\mathbf{P}_f)}$ evaluation:	5
Number of finite element analyses per α level cut evaluation of $\Pi_{(\mathbf{P}-\mathbf{P}_f)}$:	512
Total number of finite element analyses required per optimization:	76,800

In contrast, the stress distribution and buckling load response surface approximations were constructed from a total of only 752 finite element analyses. Additionally, these response surface approximations can be used in multiple optimizations without the need of performing additional finite element analyses.

On the second level, a response surface approximation of the possibility of failure as a function of the nominal values of the design variables and the level of uncertainty associated with these variables was constructed. This second level approximation was constructed to simplify the integration of the analysis code with the optimization algorithm as well as to eliminate noise in the response function, thus allowing the use of a derivative based optimization algorithm. In the present paper, the generalized reduced gradient algorithm provided with Microsoft Excel Version 7.0 was used.

The λ, β and γ design space of Table 1 was used to construct the possibility of failure response surface approximation, with numerical experiments conducted at an evenly spaced grid consisting of 11 data points in each of the λ, β and γ directions. Additionally, seven levels of uncertainty evenly spaced between ±2% and ±20% were considered, yielding a total of 2,629 data points in the design space. At each data point the possibility of failure according to each of the two failure criteria was evaluated. Two response surface approximations (one for each failure mode) were constructed using all of the data points with possibility of failure not equal to either 0 or 1. This process resulted in 499 data points for constructing the yield stress failure criterion response surface approximation and 573 data points for the buckling load constraint failure criterion response surface approximation. The resulting predicted possibility of failure is then obtained from

$$\hat{\Pi}_{(\mathbf{P}_-\mathbf{P}_f)} = \min\left(\hat{\Pi}_{\text{YieldStress}}, \hat{\Pi}_{\text{Buckling}}\right) . \quad (27)$$

It was found that a general fourth-order polynomial (70 parameters) gave accurate approximations for both failure modes. These general response surface approximations were reduced using the mixed stepwise regression procedure and the Cp statistic, with the predictive capabilities of the response surface approximations summarized in Table 4.

Table 4. Predictive capabilities of the possibility of failure response surface approximations

Model	$\mathbf{R^2}$	Adj-$\mathbf{R^2}$	RMSE [%]	PRESS [%]
Stress	4-th order model (499 data points)			
Full 70 terms	0.9988	0.9986	2.2525	2.6205
Reduced 59 terms	0.9988	0.9986	2.2371	2.5205
Buckling	4-th order model (573 data points)			
Full 70 terms	0.9982	0.9980	2.7342	3.0991
Reduced 57 terms	0.9982	0.9980	2.7118	3.0211

6 Dependence of the weight on the level of uncertainty

In order to study the dependence of the weight of the plate on the level of uncertainty associated with the design variables, different levels of uncertainty between ±2% and ±20% were considered. For each of these levels, the nondimensional cross-sectional area of the plate was minimized for an allowable possibility of failure. The allowable possibility of failure (allowable was assumed to be equal to the optimum value obtained from the fuzzy set based design problem of (26). The resulting optimization problem may be written as

minimize:

$$\tilde{A} = \frac{A}{\lambda t_0} = \frac{1}{2}(1 + 2\beta + \gamma) + \frac{\lambda}{2}(1 - 2\beta - \gamma) ,$$

subject to

$$\frac{\beta}{0.4} + 1 \geq 0 , \quad 1 - \frac{\beta}{0.4} \geq 0 , \quad \gamma \geq 0 ,$$
$$1 - \frac{\gamma + \beta}{0.4} \geq 0 , \quad \frac{\hat{\Pi}_{(\mathbf{P}-\mathbf{P}_f)}(\lambda, \beta, \gamma, u)}{\Pi_{\text{allowable}}} - 1 \geq 0 , \tag{28}$$

where u denotes the level of uncertainty associated with the design variables and $\hat{\Pi}_{(\mathbf{P}-\mathbf{P}_f)}$ denotes the predicted possibility of failure, obtained from (27).

7 Results

In order to obtain an upper limit of the weight for the fuzzy set based design, the deterministic design was evaluated first. The nondimensional cross-sectional area of the plate was minimized for a factor of safety equal to 1.5, making use of the formulation of (25). The corresponding optimum design is summarized in Table 5, where the values in parentheses are the possibility of failure values obtained from the reduced possibility of failure response surface approximations.

Table 5. Deterministic optimum (uncertainty of the design variables equal to ±5%)

Variable	Value
λ	0.6287
β	-0.4000
γ	0.0447
$\tilde{A}^*$	0.6741
Factor of safety	1.5
$\Pi_{\text{Yieldstress}}$	0.0977 (0.1181)
Π_{Buckling}	0.3411 (0.3331)

For the deterministic optimum design, both failure criteria are active. The optimum design corresponds to a plate with a change in thickness that starts at the minimum allowable distance from the left endpoint of the plate (see Fig. 4) with a very short transition zone (small γ value). Even though both failure criteria are active for the optimum design, a large difference exists between the possibility of failure for the two failure criteria, with the buckling load constraint being critical. Both the possibility of failure values obtained from the vertex method and the values obtained from the reduced possibility of failure response surface approximation are shown. The accuracy of the reduced possibility of failure response surface approximation is demonstrated since the difference between the critical predicted and calculated possibility of failure values at the optimum design is only 2.3%.

The equivalent fuzzy set based design, using the $\tilde{A}^*$ value of Table 5 as an upper limit of the weight are summarized in Table 6. Again, the values in parentheses are the possibility of failure values obtained from the reduced possibility of failure response surface approximations. The fuzzy set based optimum design corresponds to a plate where the change in thickness starts at the minimum allowable distance from the left endpoint of the plate with no transition zone (γ value equal to 0). The fuzzy set based design eliminates the weight of the ramp and uses it to thicken the thin section of the plate. The result is an increase in the stress concentration and an improvement in the buckling load of the plate. The fuzzy set based design thus attempts to equalize the possibility of failure of the two failure criteria by making the yield stress failure criterion more critical and the buckling load constraint less critical. However, for the present example problem, the design variable limits kept the possibility of failure values from becoming equal at the optimum design.

Table 6. Fuzzy optimum (uncertainty of the design variables equal to ±5%)

Variable	Value
λ	0.6379
β	-0.4000
γ	0.0000
$\tilde{A}^*$	0.6741
Factor of safety	1.4898
$\Pi_{\text{Yieldstress}}$	0.1319 (0.1262)
Π_{Buckling}	0.2788 (0.2721)

For the fuzzy set based design, the factor of safety is not much different from that of the deterministic design (only 0.7% lower), however, there is a large difference in the possibility of failure between the two designs. The possibility of failure for the fuzzy set based design is 22.3% lower than that of the deterministic design. As before, both the predicted and calculated possibility of failure values are shown in Table 6, with the difference between the critical values equal to only 2.4%.

The possibility distributions of failure for each failure mode of the optimum designs obtained from the two methods are shown graphically in Fig. 6. The possibility distributions of Fig. 6 clearly illustrate the differences in the way each method maximizes the safety measure for a given weight as discussed in Section 5.

An important tool for determining the tolerances to which a structure will be manufactured, is to know the dependence of the weight on the uncertainty associated with the geometry of the structure. The dependence of the weight of the

Fig. 6. Possibility distributions of failure for the deterministic and fuzzy set based optimum designs

structure on the level of uncertainty associated with the design variables was thus also studied. For this study, the possibility of failure was kept constant at the optimum value obtained from the fuzzy set based design (i.e., 0.2788 as summarized in Table 6), while the level of uncertainty associated with the design variables λ, β and γ was varied between $\pm 2\%$ and $\pm 20\%$. Seven levels of uncertainty, evenly distributed between $\pm 2\%$ and $\pm 20\%$, were considered. For each of these levels, the reduced possibility of failure response surface approximations and the Microsoft Excel solver was used to minimize the nondimensional cross-sectional area for the specified possibility of failure. As expected, the nondimensional cross-sectional area of the plate increased with an increase in the level of uncertainty and the results are shown graphically in Fig. 7.

Fig. 7. Nondimensional cross-sectional area associated with different level of uncertainty in the design variables

Figure 7 indicates that the increase in weight is almost linearly proportional to the increase in the uncertainty associated with the design variables. The nondimensional cross-sectional area increased by 10.7% with an 18% increase in the uncertainty associated with the design variables. Using Fig. 7 and the dependence of the manufacturing cost on the tolerance of λ, β and γ, the designer may determine what tolerance to use in manufacturing the plate.

8 Concluding remarks

It is shown that response surface approximations provide an effective approach for reducing the computational cost associated with performing a fuzzy set based design for uncertainty. The large number of computationally expensive finite element analyses required to perform the fuzzy set based design is replaced by response surface approximations that are inexpensive to evaluate. By using response surface approximations, the computational burden shifts from the optimization problem to the problem of constructing the response surface approximations. Due to the iterative nature of the design process, the fact that response surface approximations allow multiple optimizations at minimal cost should be an attractive feature to any designer. The present paper also made use of response surface approximations to simplify the integration of the analysis code and the optimization algorithm.

It was shown that for the same upper limit of the weight, the fuzzy set based design resulted in an optimum design with a possibility of failure 22.3% lower than the corresponding deterministic design. Additionally, the factor of safety of the fuzzy set based design is only 0.7% smaller than that of the deterministic design and for this example problem the fuzzy set based design is thus clearly superior. Finally, the dependence of the structural weight on the uncertainty of some key geometric parameters is presented in the form of a design chart and may be used, together with the manufacturing cost, to determine the tolerances that when manufacturing the plate. This design chart would have been very time consuming to construct if response surface approximations were not used to reduce the computational cost.

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